An Axiomatic Foundation for Epidemics on Complex Networks

Paul Loya and Adam R. Lucas

Abstract We provide a rigorous axiomatic framework to study epidemiology on complex networks. Our axioms apply to the epidemic spreading on complex networks in which there are explicit correlations among the degrees of connected vertices as described in [1]. We prove a necessary and sufficient condition for our epidemic model to have a nonzero stationary solution. We believe this is the first proof of such a general result. Moreover, under appropriate conditions we show that the time independent solution is the limit of a unique time dependent solution. We also provide a rigorous definition of the epidemic threshold, $\lambda_c := 1/\lambda_1$ with λ_1 denoting the largest positive eigenvalue of an operator T given in the axioms of our model.

1 Introduction

Understanding the mechanism behind the emergence and perseverance of infected individuals on a complex network is an important problem of interest in varied disciplines including biology, physics, social sciences and mathematics [2, 3]. From the *Susceptible-Infected-Susceptible*, (abbrev. SIS), model in epidemiology a rich mathematical theory is developing which provides insight into how a disease can spread across complex networks [4, 5]. In the SIS model on a connected undirected graph, the nodes represent individuals who are in one of two states: infected (those carrying the disease) or susceptible (those who do not have the disease yet but can catch it). The edges of the graph correspond to the contacts between individuals. Only susceptible individuals in contact with one or more infected individual may

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Paul Loya
Department of Mathematics, Binghamton University,Vestal Parkway East, Binghamton, NY 13902, U.S.A., e-mail: paul@math.binghamton.edu

Adam R. Lucas
Department of Mathematics and Computer Science, Saint Mary’s College of California, 1928 Saint Mary’s Rd., Moraga, CA 94575, U.S.A., e-mail: arl3@stmarys-ca.edu

become infected. Infected individuals can spontaneously become susceptible again. When the spreading rate of a disease is greater than the network's epidemic threshold, a nonzero fraction of the population becomes infected in the long term. For an uncorrelated network, the epidemic threshold is the quotient of the expected degree of the network and the expected degree squared [5].

How does one define the epidemic threshold when the degree of neighboring vertices in the network are correlated? In [1], they relate the presence or absence of an epidemic threshold to the eigenvalue spectra of certain connectivity matrices of the network. One of the goals of this paper is to put this observation on a firm mathematical foundation. To do this we create an axiomatic foundation for epidemics which generalizes the uncorrelated and correlated SIS models. Our epidemic model also provides a framework for studying more general interacting particle systems.

The organization of the paper is as follows. In Section 2 we will introduce the axioms of our epidemic model. In Section 3 we show that the classical correlated and uncorrelated SIS models satisfy the axioms of our model. We point out that the dynamics of our model (under appropriate assumptions) have a unique time dependent solution which converges to a stationary one. In Section 4 we prove the main result of the paper (see Theorem 2); a necessary and sufficient condition for the existence of a nonzero stationary solution for our epidemic model.

2 Axioms of the Epidemic Model

An epidemic model shall be described in terms of three primitive notions:

$$\Omega,\ \rho,\ \text{and}\ T$$

where Ω is a set, $\rho : \Omega \times [0,\infty) \to [0,1]$ is a function, and $T : L^2(\Omega) \to L^2(\Omega)$ is a continuous linear map on the Hilbert space $L^2(\Omega)$ of real-valued square integrable functions on Ω. We will first describe the intended concrete interpretation of these three primitive notions before giving the axioms.

In the study of epidemics on a graph, Ω is the set of degrees of the vertices in the graph–i.e. the number of connections the vertices make to other vertices. In this context Ω is a subset of natural numbers, $\Omega \subseteq \mathbb{N}$. The set Ω is the sample space where a probability distribution $P(k), k \in \Omega$, describes the degree (i.e. number of neighbors for each vertex) distribution of the graph.

The function $\rho : \Omega \times [0,\infty) \to [0,1]$ can be interpreted as the *epidemic state* of the system. Given $k \in \Omega$ and $t \in [0,\infty)$, the number $\rho(k,t)$ represents the probability that a degree k vertex is infected at time t.

The physical meaning of the operator $T : L^2(\Omega) \to L^2(\Omega)$ needs more explanation. Recall that a Hilbert space is a complete inner product space, where the inner product is defined by

$$\langle f,g \rangle = \int_\Omega f(k)\,g(k)\,dP(k).$$

We are in particular interested in the subset of $L^2(\Omega)$ consisting of probability functions $\varphi : \Omega \to [0,1]$ that assign to each degree $k \in \Omega$ the probability that that a degree k vertex is infected. We call such probability functions the *states* of the system.

The operator
$$T : L^2(\Omega) \to L^2(\Omega)$$
is assumed to be an non negative integral operator in the sense that there is a measurable function
$$\tau : \Omega \times \Omega \to [0,\infty)$$
such that
$$(Tf)(k) = \int_\Omega \tau(k,k')\, f(k')\, dP(k') \quad \text{for all } f \in L^2(\Omega) \text{ and } k \in \Omega. \tag{1}$$

We shall assume that τ satisfies the generalized *detailed balance condition*
$$\tau(k,k') = \tau(k',k) \quad \text{for all } \ k,k' \in \Omega.$$

This implies, in particular, that T is *symmetric* by Fubini's theorem. In the case where Ω is a finite set, as in the case when you have a finite graph, you can think of T as a symmetric matrix with non negative entries. Let $\varphi : \Omega \to [0,1]$ be a state of the system. Then the function $T\varphi : \Omega \to [0,\infty)$ is to be interpreted as follows. Given $k \in \Omega$, the value $(T\varphi)(k) \in [0,\infty)$ is a measure of the possibility that the node k can be infected.

2.1 Axioms on the epidemic model

Using the three primitive notions Ω, ρ, and T intuitively explained in the previous section we now state our axioms for an epidemic model. A system (Ω,ρ,T) shall be called an *epidemic model* if the following Axioms are satisfied.

Axiom 1: Ω is the sample space of a probability space (Ω,P), where P is a probability measure on a σ-algebra of subsets of Ω.

Axiom 2: The operator $T : L^2(\Omega) \to L^2(\Omega)$ is a compact integral operator where τ in (1) satisfies the detailed balance condition.

Axiom 3: The largest positive eigenvalue, λ_1, of T has an associated eigenvector $v : \Omega \to (0,\infty)$ that is strictly positive.

Axiom 4: The *epidemic state* of the model
$$\rho : \Omega \times [0,\infty) \to [0,1]$$
satisfies the system of nonlinear differential equations
$$\boxed{\begin{cases} \partial_t \rho(k,t) = -\rho(k,t) + \lambda(1-\rho(k,t))(T\rho)(k,t), \\ \quad \rho(k,0) = \rho_0(k), \end{cases}} \tag{2}$$

where $\rho_0 : \Omega \to [0,1]$ is the initial state of the system, assumed to be a measurable function, and where $\lambda > 0$ is called the *(effective) rate of infection*.

Axiom 5: In the case, and only in the case when, Ω is infinite, we require that for each $\lambda > 1/\lambda_1$, the inequality

$$v \le \lambda T\left(\frac{v}{1+\varepsilon v}\right)$$

holds for all $\varepsilon > 0$ sufficiently small.

3 The Classical SIS Model

Below we will see how the classical SIS models (uncorrelated and correlated) satisfy the axioms of our epidemic model.

3.1 The uncorrelated SIS model

Given a connected graph (finite or infinite) let Ω denote the set of all degrees of the nodes of the graph and suppose these nodes have the degree distribution $P(k)$. Thus, (Ω, P) is a probability space with the probability measure given by

$$P(\omega) = \sum_{k\in\omega} P(k) \quad \text{for all } \ \omega \subseteq \Omega; \tag{3}$$

here we follow the standard notation of identifying the probability measure P (on the left in (3)) with its distribution $P(k)$ (on the right in (3)). The average degree of a node is given by $\langle k \rangle = \sum_i iP(i)$. Let $\rho(k,t)$ be the density of infected degree k nodes at time t. We assume that the degrees of each node are uncorrelated and hence the probability that an edge leads to an infected node is independent of the degrees of neighboring nodes. Consider the following proposition proved elsewhere [6, 7].

Proposition 1. *The probability that an edge leads to an infected node is given by*

$$\Theta = \frac{1}{\langle k \rangle} \sum_{k'} k' P(k') \rho(k',t). \tag{4}$$

Since the higher the degree of a node, the more likely the node will be infected, it follows that given k, the number kΘ is a measure of the possibility that a node of degree k can be infected.

With this proposition at hand we can now set up the classical SIS model. First, we assume that

1. The proportion of susceptible nodes become infected at a rate proportional to $(1-\rho(k,t))k\Theta$.

Note that $1-\rho(k,t)$ represents the proportion of degree k nodes not infected (the susceptible nodes) while $k\Theta$, based on the remark concerning $k\Theta$ in Proposition 1, measures the possibility that a degree k node gets infected.

We also assume that

2. The number of infected nodes decreases at a rate proportional to the number infected nodes.

The following nonlinear differential equation describes the dynamics of our system [6]:

$$\begin{cases} \partial_t \rho(k,t) = -\rho(k,t) + \lambda(1-\rho(k,t))k\Theta(t), \\ \Theta(t) = \dfrac{1}{\langle k\rangle}\displaystyle\sum_k kP(k)\rho(k,t), \\ \rho(k,0) = \rho_0(k), \end{cases} \tag{5}$$

where ρ_0 is the initial infectivity distribution.

The uncorrelated SIS model and equation (5) is a particular case of the epidemic model, (Ω,ρ,T), given in the axioms of Section 2.1. We now verify that the axioms are satisfied.

Axiom 1: Let $\Omega \subseteq \mathbb{N}$ be the set of degrees of the graph. For example, $\Omega = \{2,5,7\}$ means that every vertex has 2, 5, or 7 connections. The sigma algebra is the power set of Ω (i.e. the set of all subsets $A \subseteq \Omega$). For $A \subseteq \Omega$ we define the probability measure $P(A) = \sum_{k\in A} P(k)$ where $P(k)$ is a probability mass function on Ω.

Axiom 2: Observe that the top equation in (5) is of the form (2) if $k\Theta = T\rho$ where

$$Tf(k) = \frac{1}{\langle k\rangle}\sum_{k'} kf(k')k'P(k')$$

for all $f \in L^2(\Omega)$. In this discrete case, integrals are really summations so we can write T as

$$Tf(k) = \int_\Omega \tau(k,k')\, f(k')\, dP(k'),$$

where $\tau(k,k') = kk'/\langle k\rangle$. By definition, for $\tau(k,k')$ to be a measurable function, the preimage of any interval in $[0,\infty)$ must be an element of the product sigma algebra of $\Omega\times\Omega$. This is trivially satisfied here since the sigma algebra is the power set of $\Omega\times\Omega$. Moreover, it is clear that τ satisfies the detailed balance condition.

Next we show that T is a compact operator. By definition of T, given any $f \in L^2(\Omega)$ we have

$$Tf(k) = \frac{k}{\langle k\rangle}\sum_{k'} f(k')k'P(k') = \frac{1}{\langle k\rangle}\langle f,v\rangle\, v(k),$$

where $\langle f, v\rangle$ is the $L^2(\Omega)$ inner product of f and v with $v : \Omega \to (0,\infty)$ the function $v(k) = k$ for all $k \in \Omega$. Hence,

$$T = \frac{1}{\langle k\rangle}\langle \cdot, v\rangle v. \tag{6}$$

To show that T is compact we must show that if f_i is a convergent sequence of functions then Tf_i has a convergent subsequence. But $\frac{1}{\langle k\rangle}\langle f_i, v\rangle$ is a bounded sequence of real numbers and so by the Bolzano-Weierstrass theorem it has a convergent subsequence. Since v is fixed, it follows that Tf_i has a convergent subsequence.

Axiom 3: From (6) we see that the image of T is the span of v. It follows that for all $f \in L^2(\Omega)$, we have $Tf = 0$ if $f \perp v$ and if $f = v$, we have

$$Tv = \lambda_1 v\,, \quad \text{where } \lambda_1 = \frac{\langle k^2\rangle}{\langle k\rangle}$$

with $\langle k^2\rangle = \langle v, v\rangle = \sum_k k^2 P(k)$, the average value of k^2. It follows that T has exactly one eigenvalue, given by λ_1 and moreover, λ_1 has an associated strictly positive eigenvector $v : \Omega \to (0,\infty)$ given by $v(k) = k$ for all $k \in \Omega$.

Axiom 4: The equations in (5) are of the form (2) if $k\Theta = T\rho$.

Axiom 5: In the case when Ω is infinite, let $\lambda > 1/\lambda_1$ and observe that

$$v \le \lambda T\left(\frac{v}{1+\varepsilon v}\right) \Longleftrightarrow v \le \frac{\lambda}{\langle k\rangle}\left\langle \frac{v}{1+\varepsilon v}, v\right\rangle v \Longleftrightarrow 1 \le \frac{\lambda}{\langle k\rangle}\left\langle \frac{v}{1+\varepsilon v}, v\right\rangle.$$

The function $g(\varepsilon) := \frac{\lambda}{\langle k\rangle}\left\langle \frac{v}{1+\varepsilon v}, v\right\rangle$ is a continuous function of $\varepsilon \in [0,\infty)$. Since $g(0) = \frac{\lambda}{\langle k\rangle}\langle v, v\rangle = \lambda\lambda_1 > 1$, by continuity $g(\varepsilon) > 1$ for all $\varepsilon > 0$ sufficiently small. This completes the verification of Axiom 5.

3.2 The correlated SIS model

Following [1] we now allow for correlations between the degrees of neighboring vertices. We henceforth assume that Ω is a connected finite graph. Let $P(k'|k)$ be the conditional probability that a vertex of degree k' is connected to a vertex of degree k. In the spirit of Proposition 1 we have the following result proved elsewhere [1, 8].

Proposition 2.

$$\Theta(k,t) = \sum_{k'} P(k'|k)\rho(k',t) \tag{7}$$

is the probability an edge originating at a degree k vertex leads to an infected vertex.

The differential equation (5) takes the following form [1]:

$$\begin{cases} \partial_t \rho(k,t) = -\rho(k,t) + \lambda(1-\rho(k,t))k\Theta(k,t), \\ \Theta(k,t) = \sum_{k'} P(k'|k)\rho(k',t), \\ \rho(k,0) = \rho_0(k), \end{cases} \tag{8}$$

where ρ_0 is the initial infectivity distribution. We now verify that the correlated SIS system satisfies our axioms.

Axiom 1: Same as in uncorrelated case but Ω now is finite.

Axiom 2: Observe that the top equation in (8) is of the form (2) if $k\Theta = T\rho$ where

$$T f(k) = \sum_{k'} kP(k'|k) f(k') \tag{9}$$

for all $f \in L^2(\Omega)$. Note that since Ω is finite, all functions $f : \Omega \to \mathbb{R}$ are in $L^2(\Omega)$. Hence, if N is the number of distinct degrees of the graph then $L^2(\Omega) \cong \mathbb{R}^N$ where we make the identification of a function $f : \Omega \to \mathbb{R}$ with the vector $(f(i_1), f(i_2), \ldots, f(i_N)) \in \mathbb{R}^N$ where $\Omega = \{i_1, \ldots, i_N\}$. With this identification, we see that written as a matrix we have $T = [T_{kk'}]$ where $T_{kk'} = kP(k'|k)$. Thus, we see that T is exactly the *correlation matrix* of [1]. Note that we can rewrite (9) as

$$T f(k) = \int_\Omega \tau(k,k')\, f(k')\, dP(k'),$$

where $\tau(k,k') = kP(k'|k)/P(k')$. As in the uncorrelated case, τ is trivially a measurable function. The detailed balance condition for physical networks is [1]:

$$kP(k'|k)P(k) = k'P(k|k')P(k').$$

This equality implies that $\tau(k,k') = \tau(k',k)$. That T is a compact operator follows from that fact that $L^2(\Omega) \cong \mathbb{R}^N$ and any linear map on $\mathbb{R}^N$ is compact by the Bolzano-Weierstrass theorem.

Axiom 3: From the connectedness of our graph it follows that the operator T is irreducible and since $kP(k'|k) \geq 0$ we see from (9) that T is nonnegative, hence by the Perron-Frobenius theorem, the largest eigenvalue λ_1 of T (is simple and) has an associated strictly positive eigenvector $v : \Omega \to (0,\infty)$ (see [9] for details).

Axiom 4: The equations in (5) are of the form (2) if $k\Theta = T\rho$.

3.3 Existence and Uniqueness theorem

The the system of nonlinear differential equations 2 does have a time dependent solution $\rho(k,t)$ which converges to the stationary one. In [8] we prove the following existence and uniqueness theorem:

Theorem 1. *With appropriate assumptions, the system of differential equations (2) has a unique solution, $\rho(k,t)$, such that*

$$\lim_{t\to\infty} \rho(k,t) = \rho_s(k),$$

where $\rho_s : \Omega \to [0,1]$ is the stationary solution to the equations (2).

As we showed in sections 3.1 and 3.2 the classical SIS models (correlated and uncorrelated) are special cases of our epidemic model. Thus, Theorem 1 applies to both of these examples. We next analyze the time-*independent* solutions.

4 The epidemic threshold for the general epidemic model

The main result of this section is the following theorem.

Theorem 2. *There exists a strictly positive stationary solution $\rho_s : \Omega \to (0,1]$ to the equations in* (2) *if and only if $\lambda > \lambda_c$ where $\lambda_c := 1/\lambda_1$ with λ_1 denoting the largest positive eigenvalue of the operator T. The number λ_c is thus referred to as the* ***epidemic threshold****.*

In particular, for the classical uncorrelated system in Section 3.1, we showed that $\lambda_1 = \langle k\rangle/\langle k^2\rangle$, so the epidemic threshold for this system is $\lambda_c = \langle k^2\rangle/\langle k\rangle$, which is the famous result in [6]. For the classical correlated system in Section 3.2, the epidemic threshold is given by $\lambda_c = 1/\lambda_1$ where λ_1 is the largest eigenvalue of the correlation matrix. This fact was also derived in [1] by another method.

4.1 Stationary solutions and a fixed point problem

Here, by a stationary solution, we mean a measurable function $\rho_s : \Omega \to [0,1]$ satisfying the first equation in (2) (omitting the initial condition requirement):[1]

$$0 = -\rho_s(k) + \lambda(1-\rho_s(k))(T\rho_s)(k), \tag{10}$$

noting that $\partial_t \rho_s(k) \equiv 0$. Solving for ρ_s we find that

$$\rho_s = \frac{\lambda T \rho_s}{1+\lambda T\rho_s}.$$

In other words, if $K = \{\text{measurable } \varphi : \Omega \to [0,1]\}$ and

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[1] Technically speaking the equality (10) only holds almost surely, that is, (10) holds except for k on a set of probability zero. However, throughout this paper, for brevity, we shall ignore sets of probability zero and write = without prefacing the equality with “almost everywhere”.

$$F : K \to K$$

is the map

$$F\varphi = \frac{\lambda T\varphi}{1+\lambda T\varphi}, \tag{11}$$

then it follows that $\rho_s : \Omega \to [0,1]$ is a stationary solution to the dynamical equation (2) if and only if $\rho_s = F\rho_s$; that is, ρ_s is a fixed point for F.

Lemma 1. *For the epidemic model, the following statements are equivalent:*

1. $\rho_s : \Omega \to [0,1]$ *is a stationary solution to* (2).
2. *The function* $F : K \to K$ *has a fixed point* $\rho_s \in K$.

Of course, $\rho_s \equiv 0$ is a stationary solution, which is uninteresting, and in an epidemic theory we are really interested in *nonvanishing* stationary solutions. In the next subsection we give necessary and sufficient conditions on λ that guarantees a nonvanishing stationary solution.

4.2 Proof of Theorem 2

Assume that there is a non vanishing stationary solution ρ_s; we shall prove that $\lambda > 1/\lambda_1$. From *(2)* of Lemma 1, we know that $\rho_s = \dfrac{\lambda T\rho_s}{1+\lambda T\rho_s}$. By assumption, ρ_s is strictly positive, so $T\rho_s$ is also strictly positive, and therefore

$$\lambda T\rho_s - \rho_s = \frac{(\lambda T\rho_s)^2}{1+\lambda T\rho_s} =: g$$

defines a strictly positive function $g : \Omega \to (0,\infty)$. Hence, if $\| \; \|$ denotes the norm on $L^2(\Omega)$, then

$$\|\rho_s\|^2 = \int \rho_s(k)^2\, dP(k) < \int (\rho_s(k)+g(k))^2\, dP(k) = \|\rho_s + g\|^2.$$

Thus, $\|\rho_s\| < \|\rho_s + g\|$. On the other hand, as $\rho_s + g = \lambda T\rho_s$, we have $\|\rho_s + g\| = \lambda\|T\rho_s\|$. Since λ_1 is the largest eigenvalue of T, it follows that

$$\|T\rho_s\| \le \lambda_1\|\rho_s\|.$$

Finally, we conclude that

$$\|\rho_s\| < \lambda\lambda_1\|\rho_s\|.$$

Dividing by $\|\rho_s\| \neq 0$ we see that $1 < \lambda\lambda_1$.

Now assume that $\lambda\lambda_1 > 1$; we want to prove there is a strictly positive function $\rho_s : \Omega \to (0,1]$ such that $\rho_s = F(\rho_s)$, where $F : K \to K$ is the map found in (11).

Step 1: To find the fixed point we construct it. Let $\rho_0 := 1$ and for $i > 0$ put

$$\rho_{i+1} = F(\rho_i).$$

(Note that $1 \in K = \{\text{measurable } \varphi : \Omega \to [0,1]\}$, so $\rho_1 := F(1) \in K$, and by recursion, all the ρ_i's belong to K.) We claim that $\{\rho_i\}$ is a non increasing sequence of functions:

$$\rho_0 \geq \rho_1 \geq \rho_2 \geq \rho_3 \geq \cdots \geq 0.$$

To prove this, we first claim that F itself is non decreasing; that is, if $0 \leq \varphi \leq \psi$, then $F(\varphi) \leq F(\psi)$. To see this, let $0 \leq \varphi \leq \psi$ and observe that

$$\begin{aligned} F(\varphi) \leq F(\psi) \Longleftrightarrow \quad & \frac{\lambda T\varphi}{1+\lambda T\varphi} \leq \frac{\lambda T\psi}{1+\lambda T\psi} \\ \Longleftrightarrow \quad & \lambda T\varphi(1+\lambda T\psi) \leq \lambda T\psi(1+\lambda T\varphi) \\ \Longleftrightarrow \quad & T\varphi \leq T\psi \quad \text{(cancel } \lambda^2 T\varphi T\psi \text{ from both sides)} \\ \Longleftrightarrow & \int \tau(k,k')\varphi(k')\,dP(k') \leq \int \tau(k,k')\psi(k')\,dP(k'), \end{aligned}$$

which is true since $\tau \geq 0$ (by assumption) and $\varphi \leq \psi$. Now it is obvious that

$$\rho_1 := \frac{\lambda T(1)}{1+\lambda T(1)} \leq 1,$$

that is, $\rho_1 \leq \rho_0$. Having just proved that F is non decreasing, we see that

$$F(\rho_1) \leq F(\rho_0)\,, \quad \text{which is to say,} \quad \rho_2 \leq \rho_1.$$

Applying F to the inequality $\rho_2 \leq \rho_1$ gives $\rho_3 \leq \rho_2$. Continuing by induction we see that $\rho_{i+1} \leq \rho_i$ and our proof is complete. It follows that for each $k \in \Omega$, the limit

$$\rho(k) := \lim_{i\to\infty} \rho_i(k)$$

exists.

Step 2: We claim that $\rho = F(\rho)$ and that $\rho(k) > 0$ for all $k \in \Omega$, which completes our proof. The fact that $\rho = F(\rho)$ is easy: Since $0 \leq \rho_i \leq 1$ for all i and the integral $\int \tau(k,k')\,1\,dP(k') = T(1)$ is finite (for almost every k), by the Lebesgue Dominated Convergence Theorem, the following interchange of limit and integral is valid:

$$\begin{aligned} \lim_{i\to\infty} T\rho_i = \lim_{i\to\infty} \int \tau(k,k')\,\rho_i(k')\,dP(k') &= \int \tau(k,k') \lim_{i\to\infty} \rho_i(k')\,dP(k') \\ &= \int \tau(k,k')\,\rho(k')\,dP(k') \\ &= T\rho. \end{aligned}$$

Therefore,

$$\rho := \lim_{i\to\infty} \rho_i = \lim_{i\to\infty} F(\rho_{i-1}) = \lim_{i\to\infty} \frac{\lambda T\rho_{i-1}}{1+\lambda T\rho_{i-1}} = \frac{\lambda T\rho}{1+\lambda T\rho} = F(\rho).$$

To prove that $\rho(k) > 0$ for all $k \in \Omega$, we need the following lemma.

Lemma 2. *Assume λ_1, the largest eigenvalue of T, has a strictly positive eigenvector $v : \Omega \to (0,\infty)$. For $\varepsilon > 0$ put*

$$\mathscr{A}_\varepsilon = \left\{ \varphi : \Omega \to [0,1]\,;\ \frac{\varepsilon v}{1+\varepsilon v} \le \varphi \le 1 \right\},$$

and let $F : K \to K$ be the map defined in (11)*. Then given $\lambda > 1/\lambda_1$, there is an $\varepsilon > 0$ such that*

$$F : \mathscr{A}_\varepsilon \to \mathscr{A}_\varepsilon.$$

Proof. We have to show that for $\varepsilon > 0$ sufficiently small, we have

$$\frac{\varepsilon v}{1+\varepsilon v} \le \frac{\lambda T\varphi}{1+\lambda T\varphi} \le 1 \quad \text{for all } \varphi \in \mathscr{A}_\varepsilon.$$

The second inequality is clear. By cross multiplying, the first inequality holds if and only if

$$\varepsilon v(1+\lambda T\varphi) \le \lambda T\varphi(1+\varepsilon v),$$

which holds if and only if (canceling $\varepsilon v \cdot \lambda T\varphi$ from both sides–which we can do since v is strictly positive) $\varepsilon v \le \lambda T\varphi$.

We note that T is monotone because it is given by $(Tf)(k) = \int t(k,k')f(k')dP(k')$, where $t(k,k')$ is nonnegative. So, if $f(k) \le g(k)$ it follows that $(Tf)(k) \le (Tg)(k)$. Now by monotonicity of T we have

$$\frac{\varepsilon v}{1+\varepsilon v} \le \varphi \quad \Longrightarrow \quad \lambda T\left(\frac{\varepsilon v}{1+\varepsilon v}\right) \le \lambda T\varphi.$$

Thus, to prove $\varepsilon v \le \lambda T\varphi$ holds, it suffices to prove that $\varepsilon v \le \lambda T\left(\frac{\varepsilon v}{1+\varepsilon v}\right)$ for $\varepsilon > 0$ sufficiently small; which is to say, we want to show that for $\varepsilon > 0$ sufficiently small,

$$v(k) \le \lambda T\left(\frac{v}{1+\varepsilon v}\right)(k) \quad \text{for all } k \in \Omega. \tag{12}$$

This is satisfied by Axiom 5 in the case when Ω is infinite. In the case when Ω is finite, this inequality also holds. Indeed, when $\varepsilon = 0$, the inequality (12) is $v(k) \le \lambda T(v)(k) = \lambda\lambda_1 v(k)$ which holds because $\lambda\lambda_1 > 1$. It follows by continuity that for each $k \in \Omega$ the inequality (12) holds for all $\varepsilon < \varepsilon_k$ for some $\varepsilon_k > 0$. Taking ε_0 to be the minimum of the ε_k's we see that for all $\varepsilon < \varepsilon_0$, the inequality (12) holds. This completes our proof.

Back to our proof, to see that $\rho > 0$, let v be a strictly positive eigenvector for T associated to the eigenvalue λ_1. Then by Lemma 2 we know that for $\varepsilon > 0$ sufficiently small, we have $F : \mathscr{A}_\varepsilon \to \mathscr{A}_\varepsilon$, which is to say,

$$\frac{\varepsilon v}{1+\varepsilon v} \le \varphi \le 1 \quad \Longrightarrow \quad \frac{\varepsilon v}{1+\varepsilon v} \le F(\varphi) \le 1.$$

Since $\rho_0 = 1$ in our sequence, we certainly have $\frac{\varepsilon \nu}{1+\varepsilon \nu} \leq \rho_0 \leq 1$. Therefore, since $\rho_1 = F(\rho_0)$, we have $\frac{\varepsilon \nu}{1+\varepsilon \nu} \leq \rho_1 \leq 1$. Hence, as $\rho_2 = F(\rho_1)$, again applying Lemma 2, we get $\frac{\varepsilon \nu}{1+\varepsilon \nu} \leq \rho_2 \leq 1$. Continuing by induction we see that $\frac{\varepsilon \nu}{1+\varepsilon \nu} \leq \rho_i \leq 1$ for all i. Taking $i \to \infty$ we obtain

$$0 < \frac{\varepsilon \nu}{1+\varepsilon \nu} \leq \rho \leq 1.$$

completing the proof.

5 Conclusions

In this paper we attempt to provide a rigorous axiomatic foundation to the study of epidemiology. The axioms of our model are given in Section 2.1. For systems that satisfy our axioms we have proved the existence of a nonzero stationary solution if and only if $\lambda > \lambda_c$ where $\lambda_c := 1/\lambda_1$ with λ_1 denoting the largest positive eigenvalue of the operator T. The number λ_c is thus referred to as the **epidemic threshold**. The classical uncorrelated and correlated SIS models are special cases of our epidemic model. Under appropriate assumptions, the nonzero stationary solution is the limit of a unique time dependant solution. Our axioms should apply to a broad range of interacting particle systems.

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