

# A one-parameter family of stationary solutions in the Susceptible-Infected-Susceptible epidemic model

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## Abstract

We study the uncorrelated Susceptible-Infected-Susceptible (SIS) model in epidemiology on top of a one parameter family of networks whose connectivity distribution range from scale free (SF) to exponential. For each network, the fraction of the population infected in the long term is a recursively defined hypergeometric function. For highly contagious diseases, with a high infection rate, the fraction of the population infected is lower when the network is SF. For less contagious diseases, the fraction of the population infected is lower when the network is exponential. This result points to an evolutionary advantage for a network being SF—namely a SF network is more resistant to the spread of a deadly disease.

*Key words:* Susceptible-Infected-Susceptible epidemic model, hypergeometric functions, scale free network, exponential network

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## 1. Introduction

From the *Susceptible-Infected-Susceptible* (SIS) model in epidemiology a rich mathematical theory is developing which provides insight into how a disease can spread across networks with different topologies [1, 2, 3]. Understanding the mathematical underpinnings of the SIS model is a prerequisite to understanding the mechanism behind the emergence and perseverance of infected individuals, a problem of interest in varied disciplines including

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biology, physics, social sciences and mathematics [4, 5]. Interestingly, hypergeometric functions, known mostly for its applications in engineering and physics, play a role in the study of epidemic models. We show here, for example, that they describe the steady state fraction of the population that is infected, as a function of the infection rate, in the SIS model.

In the SIS model on a connected undirected graph, the nodes represent individuals who are in one of two states: infected (those carrying the disease) or susceptible (those who do not have the disease yet but can catch it). The edges of the graph correspond to the contacts between individuals. Only susceptible individuals in contact with one or more infected individual may become infected. Infected individuals can spontaneously become susceptible again. When the infection rate exceeds the network's *epidemic threshold* there is a phase transition and a strictly positive fraction of the population is infected in the long term. Similar critical behavior phenomena are observed in many other physical systems including percolation [6], Ising-Potts models [7, 8], synchronization [9], reaction-diffusion processes [10], sandpiles [11] and avalanches [12], making the SIS model an important paradigm for these more complicated systems.

To better understand how network topology effects the long term distribution of infected and susceptible populations in the SIS model, we use a one parameter family of networks all having the same average connectivity. These networks are described by connectivity distributions (the probability  $P(k)$  that a node is connected to  $k$  other nodes) proportional to

$$P(k) \sim k^{-2-\gamma},$$

with  $1 \leq \gamma < \infty$  and average degree,  $\langle k \rangle = 2m$ , for a fixed positive integer  $m \in \mathbb{N}$ . At one end of the spectrum we have the *scale free* (SF) network of Barabási and Albert, which has a connectivity distribution  $P(k) \sim k^{-3}$  [5]. This kind of distribution implies that each node has a statistically significant probability of having a very large number of connections compared to the average degree of the network. Such is the case for many real world networks including metabolic networks [13], food webs [14] and links within the world wide web [15]. At the other end of the spectrum, in the limit as  $\gamma$  goes to infinity, we have the delta function where all the nodes have the same number of connections. This resembles an *exponential* network, where the connectivity distribution is described by a Poisson distribution. Examples of this kind of network are the random network of Erdős and Rényi [16] and the small world network of Watts and Strogatz [17].

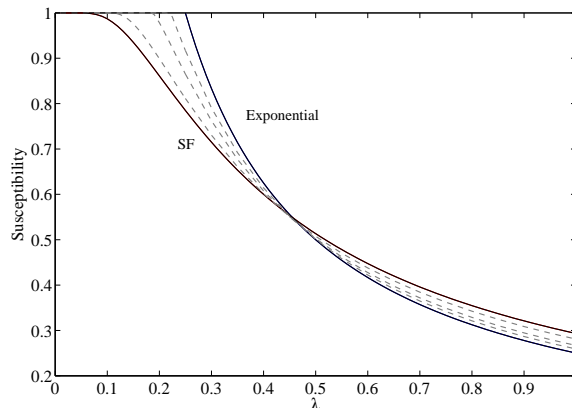


Figure 1: The susceptibility for a one parameter family of graphs  $\mathcal{G}(\gamma)$  for  $m = 2$ . The random and scale free graph intersect at  $\lambda = \frac{1}{m \log 3}$ .

Extending the work of Pastor-Satorras and Vespignani [3] we show that the steady state solution of our SIS dynamical equations (given in (3)) is a Gauss hypergeometric function [18].

**Definition 1.** Let  $a, b, c, z$  be complex numbers. The Gauss hypergeometric function is given by

$$F(a, b, c, z) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^\infty t^{c-b-1} (t+1)^{a-c} (t-z+1)^{-a} dt, \quad (1)$$

where  $\Gamma(z) = \int_0^\infty e^{-t} t^{z-1} dt$  is the gamma function,  $\text{Re}[c] > \text{Re}[b] > 0$  and  $|\arg[1-z]| < \pi$ .

Having an explicit solution allows us to study infectivity as a function of both infection rate and network topology. We show in Theorem 3.9 that infectivity for the exponential and SF topologies converge for a particular infection rate  $\lambda$ .

**Theorem 1.1.** (Theorem 3.9) Infectivity for scale free and exponential graphs coincide at a single point,  $\lambda = \frac{2}{\langle k \rangle \log 3}$ , where it takes the value  $1 - \frac{\log 3}{2}$ .

Plotting susceptibility versus  $\lambda$ , in Figure 1, we see that individuals of SF networks are more resilient against diseases with high infection rates.

Conversely, for infection rates,  $\lambda < \frac{2}{\langle k \rangle \log 3}$ , individuals of exponential like networks are more resilient.

The organization of this paper is as follows. In Section 2 we introduce the SIS model and its dynamics. In Section 3 we describe a one parameter family of recursively defined Gauss hypergeometric functions which are the steady state solution for the SIS model on different network topologies. The limiting cases are the SF and exponential models which we discuss in detail in Sections 3.2 and 3.3. Section A provides a justification of an assumption in the model that we can treat the number of connections of nodes in our network as a continuous parameter.

## 2. The Model

We consider the classical [19] SIS model

$$S \xrightarrow{\nu} I \xrightarrow{\delta} S,$$

on an undirected, connected, graph  $\mathcal{G}$  having degree (i.e. the number of edges of a nodes) distribution  $P(k)$ , and average degree

$$\langle k \rangle = \sum_i iP(i).$$

Each node in  $\mathcal{G}$  is in one of two states, susceptible,  $S$  or infected,  $I$ . At each time step, a susceptible node is infected with probability  $\nu$  if it is connected to one or more infected nodes. The nonzero probability of an infected individual to become susceptible again in the next time step is  $\delta$ . We define  $\lambda := \frac{\nu}{\delta}$  to be the effective rate of infection. Let  $s_k(t)$  and  $\rho_k(t)$  be the density of susceptible and infected degree  $k$  nodes at time  $t$  such that  $s_k(t) + \rho_k(t) = 1$ .

**Remark 2.1.**  $s_k(t)$  and  $\rho_k(t)$  are functions of the infection rate  $\lambda$ , even though our notation doesn't indicate this.

### 2.1. The classical uncorrelated SIS model

A limiting assumption in our SIS model is that the degrees of each node are uncorrelated. We assume that the probability an edge leads to an infected node is independent of the degrees of neighboring nodes. While the study of uncorrelated complex networks is a necessary first step, such correlations do occur in real systems and can have significant effects on epidemic spreading [19].

Next we define  $\Theta(\lambda, t)$  in the following Proposition whose proof is given in appendix B.  $\Theta(\lambda, t)$  is a function of the infection rate,  $\lambda$ , by Remark 2.1.

**Proposition 2.2.**

$$\Theta(\lambda, t) := \sum_k \frac{kP(k)\rho_k(t)}{\langle k \rangle} \quad (2)$$

is the probability that an edge leads to an infected node at time  $t$ .

Our assumptions about the transmission of the infection are as follows:

- i The gain of infected degree  $k$  nodes each time step is  $s_k(t)\Theta(\lambda, t)$  times the proportionality constant  $k\lambda$ .
- ii The loss of infected degree  $k$  nodes each time step is  $\rho_k(t)$ .

The following nonlinear differential equation [3], describes the dynamics of our system:

$$\begin{cases} \partial_t \rho_k(t) = -\rho_k(t) + k\lambda(1 - \rho_k(t))\Theta(\lambda, t), \\ \Theta(\lambda, t) = \frac{1}{\langle k \rangle} \sum_k k P(k)\rho_k(t), \\ \rho_k(0) = \rho_k^0, \end{cases} \quad (3)$$

where  $\rho_k^0$  is the initial infectivity distribution.

The easiest solution of (3) is the so-called stationary (time independent) solution. To find this solution, we assume that  $\rho_k(t) = \rho_k$  is independent of time in (3) and solve for  $\rho_k$ ; we then obtain

$$\rho_k = \frac{k\lambda\Theta(\lambda)}{1 + k\lambda\Theta(\lambda)}, \quad \text{where } \Theta(\lambda) = \frac{1}{\langle k \rangle} \sum_k kP(k)\rho_k. \quad (4)$$

The fact that there exists a solution  $\rho_k$  satisfying (4) is proved in Theorem 3.3.

A natural question is:

*Given an initial infectivity  $\rho_k^0$  at time  $t = 0$ , does the differential equation (3) have a (unique) solution  $\rho_k(t)$  such that  $\rho_k(0) = \rho_k^0$ , and such that*

$$\lim_{t \rightarrow \infty} \rho_k(t) = \rho_k,$$

*the stationary solution?*

This question is highly nontrivial not only because of the non-linearity of the system (3) but also because of the non-trivial dependence of  $\rho_k(t)$  on knowing all other  $\rho_*(t)$ 's. Indeed, notice that  $\Theta(\lambda, t)$  involves knowing  $\rho_k(t)$  for all  $k$ , so for fixed  $k$ , the partial derivative  $\partial_t \rho_k(t)$ , since it's written in terms of  $\Theta(\lambda, t)$ , involves knowing *all the other*  $\rho_*(t)$ 's. The answer is affirmative and discussed in a forthcoming paper [20] as a corollary of a much more general result which also encompasses correlation.

### 3. A one-parameter family for the continuum SIS model

#### 3.1. The one-parameter family

Henceforth we will make the assumption that the degrees of our network,  $\mathcal{G}$ , can be treated as a continuous variable (so  $k \in \mathbb{R}$ ). In Section A we will justify this. We call the SIS model with this continuity assumption the continuum SIS model.

We define a one parameter family of graphs  $\mathcal{G}(\gamma)$ ,  $\gamma \in [1, \infty)$ , having degree distribution given by

$$P(k) = (1 + \gamma)\alpha(\gamma)^{1+\gamma}k^{-2-\gamma}, \quad k \geq \alpha(\gamma), \quad (5)$$

where  $\alpha(\gamma) = \frac{2m\gamma}{1+\gamma}$ , and  $m$  is a positive integer. To simplify notation we will henceforth write  $\alpha$  instead of  $\alpha(\gamma)$ . Equation (2.2) now becomes

$$\Theta(\lambda) = \int_{\alpha}^{\infty} \frac{k\rho_k P(k)}{\langle k \rangle} dk. \quad (6)$$

We have

$$\rho_k = \frac{k\lambda\Theta(\lambda)}{1 + k\lambda\Theta(\lambda)}, \quad (7)$$

and *infectivity*,

$$\rho = \int_{\alpha}^{\infty} \rho_k P(k) dk. \quad (8)$$

One can easily confirm that

$$\langle k \rangle = \int_{\alpha}^{\infty} kP(k)dk = 2m.$$

Thus, the average degree of a node for our family is always  $2m$ .  $\mathcal{G}(1)$  consists of a scale free graph and we will show in Section 3.2 that  $\lim_{\gamma \rightarrow \infty} \mathcal{G}(\gamma)$  consists of a random graph having an exponential degree distribution.

In the following Lemma we relate the infectivity directly in terms of  $\Theta(\lambda)$ .

**Lemma 3.1.** *For the uncorrelated epidemic model, we have*

$$\rho = \langle k \rangle \lambda \Theta(\lambda) (1 - \Theta(\lambda)). \quad (9)$$

PROOF. Recalling from (4) that  $\rho_k = \frac{k\lambda\Theta(\lambda)}{1+k\lambda\Theta(\lambda)}$ , we see that

$$\rho_k + k\rho_k\lambda\Theta(\lambda) = k\lambda\Theta(\lambda).$$

Multiplying both sides of this equation by  $P(k)$ , integrating, and using equations (6) and (8), we obtain

$$\rho + \langle k \rangle \Theta(\lambda) \cdot \lambda \Theta(\lambda) = \langle k \rangle \lambda \Theta(\lambda).$$

Solving for  $\rho$  we get our result.  $\square$

We denote the epidemic threshold for our SIS model by  $\lambda_c$ . For an uncorrelated network, the epidemic threshold is the quotient of the expected degree of the network and the expected degree squared [19]. The following is another expression for  $\lambda_c$ .

**Lemma 3.2.** *For the uncorrelated epidemic model, we have*

$$\frac{\lambda - \lambda_c}{\lambda} = \int_{\alpha}^{\infty} \frac{k^2 P(k) \rho_k}{\langle k^2 \rangle} dk, \quad \text{where } \rho_k = \frac{k\lambda\Theta(\lambda)}{1+k\lambda\Theta(\lambda)}. \quad (10)$$

PROOF. From equations (6) and (7) we obtain

$$\Theta(\lambda) = \frac{\lambda\Theta(\lambda)}{\langle k \rangle} \int_{\alpha}^{\infty} \frac{k^2 P(k)}{1+k\lambda\Theta(\lambda)} dk.$$

Canceling the  $\Theta(\lambda)$ 's and using the identity,  $\frac{1}{1+k\lambda\Theta(\lambda)} = 1 - \rho_k$ , we get

$$1 = \frac{\lambda}{\langle k \rangle} \int_{\alpha}^{\infty} k^2 (1 - \rho_k) dP.$$

Writing  $\lambda_c = \frac{\langle k \rangle}{\langle k^2 \rangle}$  and following some algebra we get

$$\frac{\lambda - \lambda_c}{\lambda} = \int_{\alpha}^{\infty} \frac{k^2 \rho_k}{\langle k^2 \rangle} dP$$

as claimed.  $\square$

**Theorem 3.3.** For the continuum SIS model of the family of graphs  $\mathcal{G}(\gamma)$  for  $\gamma \in [1, \infty)$ , the functions  $\Theta(\lambda)$  and  $\rho$  can be expressed as the following Gauss hypergeometric functions:

(1)  $\Theta(\lambda)$  is given in terms of the Gauss hypergeometric function through

$$\Theta(\lambda) = F(1, \gamma, \gamma + 1, -(\alpha\lambda\Theta(\lambda))^{-1}) \quad (11)$$

(2) We can also express  $\lambda - \lambda_c$  in terms of  $\Theta(\lambda)$  via

$$\frac{\lambda - \lambda_c}{\lambda} = F(1, \gamma - 1, \gamma, -(\alpha\lambda\Theta(\lambda))^{-1}).$$

(3) Infectivity is given by

$$\rho(\lambda) = F(1, \gamma + 1, \gamma + 2, -(\alpha\lambda\Theta(\lambda))^{-1}). \quad (12)$$

PROOF. Substituting equations (5) and (7) into equation (6) we get the expression,

$$\begin{aligned} \Theta(\lambda) &= \frac{1}{2m} \int_{\alpha}^{\infty} k(1 + \gamma)\alpha^{1+\gamma}k^{-2-\gamma} \frac{k\lambda\Theta(\lambda)}{1 + \lambda k\Theta(\lambda)} dk \\ &= \frac{\lambda\Theta(\lambda)}{2m} (1 + \gamma)\alpha^{1+\gamma} \int_{\alpha}^{\infty} k^{-\gamma}(1 + \lambda k\Theta(\lambda))^{-1} dk. \end{aligned}$$

Making the change of variables  $k = \alpha(t + 1)$  and  $dk = \alpha dt$ , we have

$$\Theta(\lambda) = \frac{\lambda\Theta(\lambda)}{2m} (1 + \gamma)\alpha^{1+\gamma} \int_0^{\infty} (\alpha(t + 1))^{-\gamma} (1 + (t + 1)\alpha\lambda\Theta(\lambda))^{-1} \alpha dt.$$

Following some algebra this leads to the equation

$$\begin{aligned} \frac{(1 + \gamma)\alpha}{2m} \int_0^{\infty} t^0(t + 1)^{-\gamma} (t + 1 - (-\alpha\lambda\Theta(\lambda))^{-1})^{-1} dt \\ = \gamma \int_0^{\infty} t^0(t + 1)^{-\gamma} (t + 1 - (-\alpha\lambda\Theta(\lambda))^{-1})^{-1} dt. \end{aligned}$$

We notice that this integral is a Gauss hypergeometric function of the form given in Definition 1, when we let  $a = 1$ ,  $b = \gamma$ ,  $c = \gamma + 1$  and  $z = -(\alpha\lambda\Theta(\lambda))^{-1}$ . We see that

$$\int_0^{\infty} t^0(t + 1)^{-\gamma} (t + 1 - (-\alpha\lambda\Theta(\lambda))^{-1})^{-1} dt = \frac{1}{\gamma} F(1, \gamma, \gamma + 1, -(\alpha\lambda\Theta(\lambda))^{-1}).$$

The first claim then follows easily.

We can prove the second claim in a couple ways: First, by direct integration and second using well-known identities of hypergeometric functions. We present both methods. Recall from Lemma 3.2 for the general system, we have for  $\lambda > \lambda_c$ ,

$$\frac{\lambda - \lambda_c}{\lambda} = \int_{\alpha}^{\infty} \frac{k^2 \rho_k P(k)}{\langle k^2 \rangle} dk, \quad \text{where } \rho_k = \frac{k \lambda \Theta(\lambda)}{1 + k \lambda \Theta(\lambda)}.$$

Applying this formula for our continuum model, we have

$$\frac{\lambda - \lambda_c}{\lambda} = \frac{\lambda^2 \Theta(\lambda)}{\langle k \rangle} \int_{\alpha}^{\infty} \frac{k^3 p(k)}{1 + k \lambda \Theta(\lambda)} dk.$$

The derivation of the proof from this point closely follows the method of we used in Statement (1). To prove Statement (2) using identities, recall that<sup>1</sup>

$$F(1, \gamma - 1, \gamma, z) = \frac{z}{\gamma} F(1, \gamma, \gamma + 1, z) + 1.$$

Now putting  $z = -(\alpha \lambda \Theta(\lambda))^{-1}$  and using that  $\Theta(\lambda) = F(1, \gamma, \gamma + 1, -(\alpha \lambda \Theta(\lambda))^{-1})$ , after some algebra we get Statement (2).

Finally, to prove Statement (3) we could use the formula  $\rho = \langle k \rangle \lambda \Theta(\lambda) (1 - \Theta(\lambda))$  from Lemma 3.1, the hypergeometric formula in Statement (1) for  $\Theta(\lambda)$ , and identities for hypergeometric functions. Alternatively, we can determine  $\rho$  by direct integration via equation (8). We then follow the derivation of the hypergeometric formula for  $\Theta(\lambda)$  in Statement (1). Either method proves Statement (3).  $\square$

Notice that equation (11) has  $\Theta(\lambda)$  on both the left and right hand side. Hence this is a recursively defined Gauss hypergeometric function. Similarly the right hand side of equation (11) can be written in terms of  $\rho_k$ .

### 3.2. The Exponential Case

Random graphs of the kind studied in the small world model of Watts-Strogatz and the Erdős and Rényi model, have exponential degree distributions peaked at an average value  $\langle k \rangle$  that decays exponentially fast for

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<sup>1</sup><http://functions.wolfram.com/10.06.17.0003.01>

$k$  away from  $\langle k \rangle$ . We consider an extreme case where our exponential distribution is the delta function where  $P(k) = 1$  for  $k = \langle k \rangle$  and zero otherwise. In this section we prove that when  $\gamma \rightarrow \infty$ , the infectivity approaches that of the exponential model

$$\rho = \begin{cases} 1 - \frac{1}{2m\lambda} & \text{for } \lambda > \lambda_c = \frac{1}{\langle k \rangle} = \frac{1}{2m} \\ 0 & \text{for } \lambda \leq \lambda_c \end{cases}$$

Thus, equation (5) gives a one-parameter family from the SF model,  $\gamma = 1$ , to the exponential model,  $\gamma = \infty$ . To prove this result we need the following.

**Lemma 3.4.** *For the Gauss hypergeometric function, we have the identity*

$$\lim_{\gamma \rightarrow \infty} F(1, \gamma, \gamma + 1, z) = \frac{1}{1 - z}.$$

PROOF. Using Definition 1 for the Gauss hypergeometric function we have

$$\begin{aligned} F(1, \gamma, \gamma + 1, z) &= \frac{\Gamma(\gamma + 1)}{\Gamma(\gamma) \Gamma(1)} \int_0^\infty (t + 1)^{-\gamma} (t - z + 1)^{-1} dt \\ &= \gamma \int_0^\infty (t + 1)^{-\gamma} (t - z + 1)^{-1} dt, \end{aligned}$$

where we used that  $\Gamma(1) = 1$  and  $\Gamma(\gamma + 1) = \gamma \Gamma(\gamma)$ . Now observe that

$$\gamma(t + 1)^{-\gamma} = -(t + 1) \frac{d}{dt} (t + 1)^{-\gamma},$$

so

$$F(1, \gamma, \gamma + 1, z) = - \int_0^\infty \left( \frac{d}{dt} (t + 1)^{-\gamma} \right) (t + 1) (t - z + 1)^{-1} dt.$$

Integrating by parts, we obtain

$$\begin{aligned} F(1, \gamma, \gamma + 1, z) &= -(t + 1)^{-\gamma} (t + 1) (t - z + 1)^{-1} \Big|_{t=0}^{t=\infty} \\ &\quad - \int_0^\infty (t + 1)^{-\gamma} \frac{d}{dt} \left( (t + 1) (t - z + 1)^{-1} \right) dt. \end{aligned}$$

The first term on the right (the evaluated term  $\Big|_{t=0}^{t=\infty}$ ) is equal to  $\frac{1}{1-z}$  while the second term (the integral)  $\rightarrow 0$  as  $\gamma \rightarrow \infty$  by the Lebesgue Dominated Convergence Theorem. This proves our Lemma.  $\square$

**Theorem 3.5.** *When  $\gamma = \infty$ , we have*

$$\rho = \begin{cases} 1 - \frac{1}{2m\lambda} & \text{for } \lambda > \lambda_c = \frac{1}{\langle k \rangle} = \frac{1}{2m} \\ 0 & \text{for } \lambda \leq \lambda_c \end{cases} \quad (13)$$

PROOF. Using equation (11) and Lemma 3.4 we get

$$\begin{aligned} \Theta(\lambda) &= \lim_{\gamma \rightarrow \infty} F(1, \gamma, \gamma + 1, -(\alpha\lambda\Theta(\lambda))^{-1}) = \frac{1}{1 - (-(\alpha\lambda\Theta(\lambda))^{-1})} \\ &= \frac{1}{1 + \frac{1}{\alpha\lambda\Theta(\lambda)}}. \end{aligned}$$

Solving for  $\Theta(\lambda)$ , we obtain

$$\Theta(\lambda) = 1 - \frac{1}{2m\lambda}. \quad (14)$$

Therefore,

$$\rho = \langle k \rangle \lambda \Theta(\lambda)(1 - \Theta(\lambda)) = 2m\lambda \left(1 - \frac{1}{2m\lambda}\right) \cdot \frac{1}{2m\lambda} = 1 - \frac{1}{2m\lambda},$$

which proves the theorem.  $\square$

The exponential model is really the “delta function model,” but we shall see there is good reason to call the model an “exponential model”. Let  $P$  be the delta function at  $\omega$ ; that is,  $P(k) = 1$  if  $k = \omega$  and  $P(k) = 0$  otherwise. In this case we have,

$$\Theta(\lambda) = \sum_k \frac{kP(k)\rho_k(t)}{\langle k \rangle} = \rho_\omega(t),$$

so equation (3) is

$$\begin{cases} \partial_t \rho_k(t) = -\rho_k(t) + k\lambda(1 - \rho_k(t))\Theta(\lambda, t), \\ \Theta(\lambda, t) = \rho_\omega(t), \\ \rho_k(0) = \rho_k^0. \end{cases} \quad (15)$$

Thus, to find the solution  $\rho_k(t)$  we first determine the solution when  $k = \omega$  and then we derive the solution  $\rho_k(t)$  for general  $k$ . Dropping the notation  $\omega$  for brevity, we are interested in first solving the system

$$\begin{cases} \frac{d\rho}{dt} = -\rho(t) + k\lambda(1 - \rho(t))\rho(t), \\ \rho(0) = \rho^0. \end{cases} \quad (16)$$

As the next Lemma (whose proof is given in appendix B) shows, the solution to this system explicitly involves the exponential functions and hence the term “exponential model” is quite appropriate.

**Lemma 3.6.** *The system (16) has a unique solution, which is given by*

$$\rho(t) = \begin{cases} \frac{\rho^0 \rho^1}{\rho^0 + (\rho^1 - \rho^0) e^{-(k\lambda-1)t}} & \text{if } k\lambda \neq 1 \\ \frac{\rho^0}{t\rho^0 + 1} & \text{if } k\lambda = 1, \end{cases}$$

where  $\rho^1 = 1 - \frac{1}{k\lambda}$ .

We can now solve the system (15) for general  $k$ .

**Theorem 3.7.** *The system (15) has a unique solution,*

$$\rho_k(t) = 1 - \left[ e^{-(t+k\lambda \int_0^t \rho(s) ds)} (1 - \rho^0(k)) + \int_0^t e^{-(t-s)-k\lambda \int_s^t \rho(r) dr} ds \right],$$

where  $\rho(t)$  on the right hand side is the solution given in Lemma 3.6. Moreover, for  $\lambda > \lambda_c$ , we have

$$\lim_{t \rightarrow \infty} \rho_k(t) = \rho_k.$$

PROOF. Putting

$$T(t) = \int_0^t \Theta(s) ds,$$

with  $\Theta(s) = \rho_\omega(s)$ , and

$$f(k, t) = e^{t+k\lambda T(t)} \rho(k, t),$$

we see that

$$\partial_t f(k, t) = (1 + k\lambda \Theta(t)) e^{t+k\lambda T(t)} \rho_k(t) + e^{t+k\lambda T(t)} \partial_t \rho_k(t).$$

Using the differential equation (15), this equality simplifies to

$$\partial_t f(k, t) = k\lambda \Theta(t) e^{t+k\lambda T(t)} = e^t \partial_t e^{k\lambda T(t)},$$

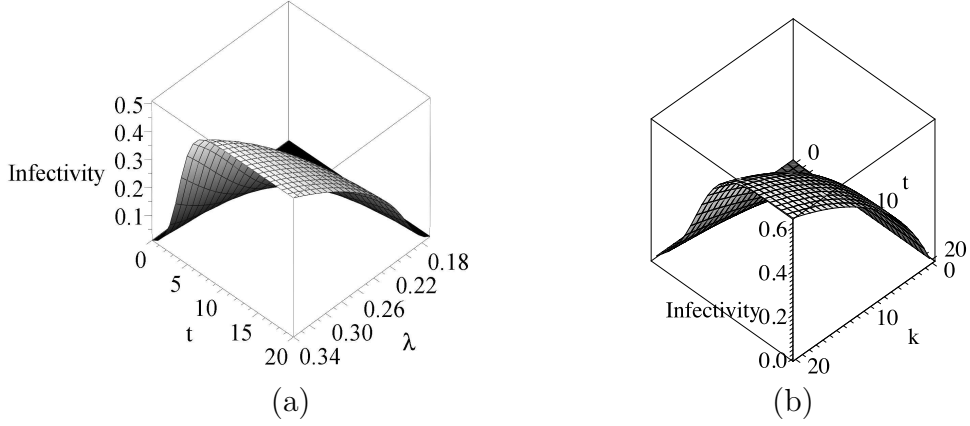


Figure 2: A graph of infectivity,  $\rho_k(t)$ , given in Theorem 3.7 for the uncorrelated epidemic model with  $\omega = 6$  and  $\rho_k^0 = .01$ . Figure (a) fixes  $k = \omega$  and plots the time evolutions for  $\lambda > \lambda_c$ . Figure (b) sets  $\lambda = .25$  and plots the time evolution for varying  $k$ .

Integrating both sides from 0 to  $t$  and integrating by parts, we obtain

$$f(k, t) - f(k, 0) = e^{t+k\lambda T(t)} - 1 - \int_0^t e^{s+k\lambda T(s)} ds.$$

Multiplying both sides of this equality by  $e^{-(t+k\lambda T(t))}$  and using that

$$f(k, t) = e^{t+k\lambda T(t)} \rho_k(t),$$

we get the desired formula for  $\rho_k(t)$ .

We shall leave the last statement concerning  $\lim_{t \rightarrow \infty} \rho_k(t)$  to the interested reader.  $\square$

Figure 2 shows the graph of infectivity,  $\rho_k(t)$ , given in Theorem 3.7 for the uncorrelated epidemic model with  $\omega = 6$  and  $\lambda_c = \frac{1}{6}$ . Figure (a) shows the special case where  $k = \omega$ . Here we see that infectivity increases with time for  $\lambda > \lambda_c$ . Figure (b) sets  $\lambda = .25$  and shows that infectivity increases with time and with increasing  $k$ . Note however that because  $P(k) = 0$  for  $k \neq \omega$ , only the infectivity at  $k = \omega$  impacts the average infectivity.

### 3.3. The Scale Free Case

Using equation (5) when  $\gamma = 1$  we get

$$P(k) = \frac{2m^2}{k^3}, \quad (17)$$

for  $k \geq m$ , which is the degree distribution for a scale free graph. By substituting equations (7) and (17) into equation (6) we get

$$\Theta(\lambda) = \frac{1}{\langle k \rangle} \int_m^\infty \frac{2km^2}{k^3} \frac{k\lambda\Theta(\lambda)}{1 + k\lambda\Theta(\lambda)}.$$

Canceling  $\Theta(\lambda)$  and using the identity  $\langle k \rangle = 2m$  we get

$$\frac{1}{m\lambda} = \int_m^\infty \frac{1}{k(1 + k\lambda\Theta(\lambda))} dk.$$

Using partial fractions and integrating this gives

$$\frac{1}{m\lambda} = -\log \left( \frac{m\lambda\Theta(\lambda)}{1 + m\lambda\Theta(\lambda)} \right).$$

Taking the exponential on both sides gives

$$e^{\frac{-1}{m\lambda}} = \frac{m\lambda\Theta(\lambda)}{1 + m\lambda\Theta(\lambda)}.$$

Solving for  $\Theta(\lambda)$  this leads to the expression

$$\Theta(\lambda) = \frac{e^{\frac{-1}{m\lambda}}}{m\lambda(1 - e^{\frac{-1}{m\lambda}})}. \quad (18)$$

Then using Lemma 3.1 we get the following.

**Theorem 3.8.** *Let  $\mathcal{G}$  be a scale free graph with degree distribution  $P(k) = \frac{2m^2}{k^3}$ , then the density of the nodes infected at equilibrium is*

$$\rho(\lambda) = \frac{2e^{\frac{-1}{m\lambda}}}{1 - e^{\frac{-1}{m\lambda}}} \left( 1 - \frac{e^{\frac{-1}{m\lambda}}}{m\lambda(1 - e^{\frac{-1}{m\lambda}})} \right).$$

It is of interest to find the  $\lambda$  value(s) where the scale free and the exponential  $\rho(\lambda)$  coincide. To establish this we set  $\rho_{\text{sf}} = \rho_{\text{exp}}$  and apply equation (9) to give  $\theta_{\text{sf}}(1 - \theta_{\text{sf}}) = \theta_{\text{exp}}(1 - \theta_{\text{exp}})$ . This implies that  $\theta_{\text{sf}} = \theta_{\text{exp}}$  or  $\theta_{\text{sf}} = 1 - \theta_{\text{exp}}$ . By equations (14) and (18) we have

$$\frac{e^{\frac{-1}{m\lambda}}}{m\lambda(1 - e^{\frac{-1}{m\lambda}})} = 1 - \frac{1}{2m\lambda}.$$

Solving for  $\lambda$  we get  $\lambda = \frac{1}{m \log 3}$ . Solving the equation  $\theta_{\text{sf}} = 1 - \theta_{\text{exp}}$  also gives  $\lambda = \frac{1}{m \log 3}$ . Setting  $m = \frac{\langle k \rangle}{2}$  and evaluating  $\rho(\lambda)$  at  $\lambda = \frac{1}{m \log 3}$  using equation (13) we have the following.

**Theorem 3.9.** *Infectivity for scale free and exponential graphs coincide at a single point,  $\lambda = \frac{2}{\langle k \rangle_{\log 3}}$ , where it takes the value  $1 - \frac{\log 3}{2}$ .*

In Figure 1 we plot susceptibility versus  $\lambda$  for our one parameter family, with  $\langle k \rangle = 4$ , using  $\rho$  given in Theorems 3.3, 3.5, and 3.8. Although  $\lambda$  is shown only between zero and one it is straightforward to confirm from the equations for  $\rho$  that the susceptibility goes to zero as  $\lambda$  goes to infinity. We see that  $\lambda_c \approx 0$  for our SF network (in the limit of an infinite size network,  $\lambda_c = 0$  [21]). A plausible physical explanation is that if one of the hubs (i.e. vertices with high connectivity) is initially infected, then the disease will spread even for a very low infection rate. This isn't the case for a homogeneous network where there are no hubs. In that case it takes a much higher infection rate for the disease to pass through the network. For our exponential network,  $\lambda_c = \frac{1}{4}$  and for our one parameter family  $0 \leq \lambda_c \leq \frac{1}{4}$ .

We can see from Figure 1 that for  $\lambda < \frac{1}{m \log 3}$  the density of susceptible nodes is lower for scale free graphs than for exponential graphs, however for  $\lambda > \frac{1}{m \log 3}$  the trend is reversed. In particular, the slope of the one parameter family gets increasingly negative, at  $\lambda = \frac{1}{m \log 3}$ , as  $\gamma$  increases from 1 to  $\infty$ . Since the average degree for each network is identical,  $\langle k \rangle = 2m$ , the explanation must lie in the degree of heterogeneity of the network. In SF networks increasing  $\lambda$  results in only a modest change in infectivity compared with an exponential network. This is because in a SF network most of the nodes have a connectivity lower than the average degree and so even with a high infectivity rate the transmission of the disease is slower than in an exponential network where all of the nodes have the average number of connections.

We conclude that to prevent the long term spread of disease that it is advantageous for a network to be scale free for highly infectious diseases (i.e. with  $\lambda > \frac{1}{m \log 3}$ ) and better for a network to be random (or exponential) for less infectious diseases. Numerical calculations indicate that the one parameter family shown in Figure 1 intersect for  $\lambda$  values within the interval  $\frac{1}{m \log 3} \pm 10^{-3}$ .

#### 4. Conclusions

The SIS model on top of a one parameter family of networks having probability distributions between scale free and the exponential distribution is examined. Gauss hypergeometric functions arise in expressions for,  $\Theta(\lambda)$ ,

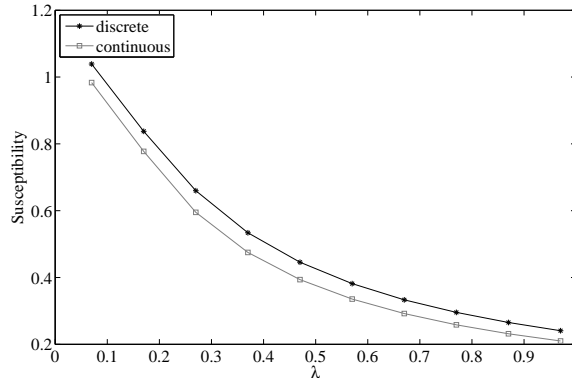


Figure 3: Comparison of susceptibility,  $1 - \rho$ , for a discrete and continuous distribution for  $m = 3$ ,  $\gamma = 1$ .

the probability that an edge leads to an infected node, the infectivity,  $\rho$ , and the epidemic threshold. These functions allow us to compare the long term trends in infectivity. Infectivity for SF and exponential networks intersect for infection rate  $\lambda = \frac{2}{\langle k \rangle \log 3}$ . Contagious diseases will infect a larger percentage of the population when spread on a SF network as compared to an exponential one. For less contagious diseases, the opposite is true.

### A. Discrete versus Continuous Distributions

We assume in the continuum SIS model that the degree of a vertex in our graph can be treated as a continuous variable. Here we justify this assumption.

Let  $\alpha = \frac{2m\gamma}{1+\gamma}$ . We define the probability mass function

$$Q(j) = \frac{(j + \alpha)^{-2-\gamma}}{\sum_{i=0}^{\infty} (i + \alpha)^{-2-\gamma}},$$

where  $j$  is a non-negative integer. Probability  $Q(j)$  is the discrete version of the continuous probability function

$$P(k) = (1 + \gamma)\alpha(\gamma)^{1+\gamma}k^{-2-\gamma}, \quad k \geq \alpha(\gamma), \quad (19)$$

given in equation (5). To see this substitute  $(j + \alpha)$  for the continuous parameter  $k$  in (19) and divide by  $(1 + \gamma)\alpha^{1+\gamma} \sum_{i=0}^{\infty} (i + \alpha)^{-2-\gamma}$  to normalize

$Q(j)$ . Our differential equation is

$$\partial_t \rho_j(t) = -\rho_j(t) + \lambda(j + \alpha)[1 - \rho_j(t)]\Theta(\lambda, t),$$

where  $\Theta(\lambda, t)$  now is

$$\Theta(\lambda, t) = \sum_{j=0}^{\infty} \frac{(j + \alpha)Q(j)\rho_j(t)}{\langle j + \alpha \rangle}$$

and

$$\langle j + \alpha \rangle = \sum_{j=0}^{\infty} (j + \alpha)Q(j).$$

Analogous to equation (4) we have the time independent stationary solution,

$$\rho_j = \frac{(j + \alpha)\lambda\Theta(\lambda)}{1 + (j + \alpha)\lambda\Theta(\lambda)}, \quad \text{where } \Theta(\lambda) = \sum_{j=0}^{\infty} \frac{(j + \alpha)Q(j)\rho_j}{\langle j + \alpha \rangle}. \quad (20)$$

Next we use numerical methods to compute  $\Theta(\lambda)$  in equation (20). Substituting  $\rho_k$  in  $\Theta(\lambda)$  in equation (20) and canceling  $\Theta(\lambda)$ , we see that  $\Theta(\lambda)$  satisfies

$$1 = \frac{\lambda}{\langle j + \alpha \rangle} \sum_{j=0}^{\infty} \frac{(j + \alpha)^2 Q(j)}{1 + (j + \alpha)\lambda\Theta(\lambda)}.$$

We define

$$F(x) = \frac{\lambda}{\langle j + \alpha \rangle} \sum_{j=0}^{\infty} \frac{(j + \alpha)^2 Q(j)}{1 + (j + \alpha)\lambda x},$$

and observe that  $F(0) = \frac{\lambda}{\lambda_c}$ , so  $F(0) > 1$  for  $\lambda > \lambda_c$ . Furthermore,

$$\begin{aligned} F(1) &= \frac{\lambda}{\langle j + \alpha \rangle} \sum_{j=0}^{\infty} \frac{(j + \alpha)^2 Q(j)}{1 + (j + \alpha)\lambda} \\ &< \frac{\lambda}{\langle j + \alpha \rangle} \sum_{j=0}^{\infty} \frac{(j + \alpha)^2 Q(j)}{(j + \alpha)\lambda} = 1, \end{aligned}$$

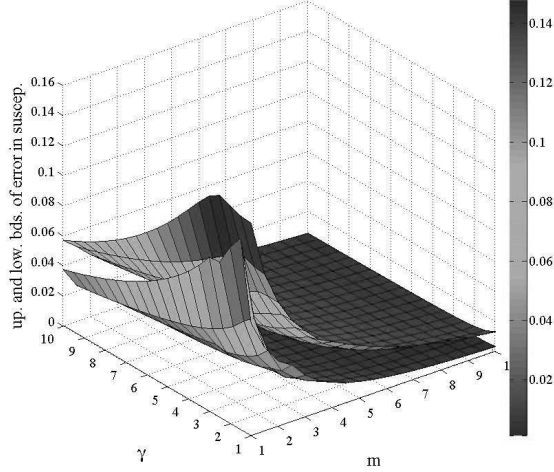


Figure 4: For each fixed  $m$  and  $\gamma$  we find bounds on the integral approximation of infectivity, (i.e. max and min of the set  $\{\rho_C(\lambda) - \rho_D(\lambda) \text{ for } 0 \leq \lambda \leq 1\}$ ). Note that infectivity using the continuous probability distribution is an under-approximation. The error is maximal, with a value of .1475 for  $m = 3$  and  $g = 1$  but quickly decreases.

since  $\frac{1}{1+(j+\alpha)\lambda} < \frac{1}{(j+\alpha)\lambda}$ .  $F(x)$  is a strictly decreasing function between  $x = 0$  and  $x = 1$ . Using Newton's method, we reiterate the sequence

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

to determine a root of  $f(x) = F(x) - 1$  between  $x = 0$  and  $x = 1$ .

Using Newton's method, as described above, with,  $P(k)$  or  $Q(k)$ , we find  $\Theta(\lambda)$  given in equations 6 and 20 respectively. Using equation (9) with the continuous and discrete probability distribution, we find the continuous and discrete susceptibility,  $1 - \rho_C$  and  $1 - \rho_D$ , respectively.

Figure 3 compares the susceptibility using the continuous and discrete probability functions,  $P(k)$  and  $Q(k)$  for  $m = 3$  and  $\gamma = 1$ . The continuous susceptibility,  $1 - \rho_C$ , is an underestimation of the discrete susceptibility,  $1 - \rho_D$ . Figure 4 shows bounds on the error of calculating susceptibility using the continuous probability distribution  $P(k)$ . For each  $m$  and  $\gamma$  value, we are able to produce a plot as in Figure 3 and find the max and min difference between the two curves for  $0 \leq \lambda \leq 1$ . Figure 4 indicates that Figure 3

represents a worst case scenario with  $.1033 \leq \rho_D(\lambda) - \rho_C(\lambda) \leq .1475$ . The continuous approximation of susceptibility quickly approaches the discrete value as  $m$  or  $\gamma$  increases. We conclude that the integral approximation method used in this paper is valid, especially for larger  $m$  and  $\gamma$  values.

## B. Additional proofs

### Proposition B.1.

$$\Theta(\lambda, t) := \sum_k \frac{kP(k)\rho_k(t)}{\langle k \rangle}$$

*is the probability that an edge leads to an infected node at time  $t$ .*

PROOF. Consider a graph,  $\mathcal{G}$  with  $N$  nodes. Cut every edge in the graph at its midpoint and consider the set,  $S$ , of all half edges. This removes all connectivity correlations in the network. Let  $E$  be the subset of half edges connected to a vertex of degree  $k$ . Then the size of these sets are,

$$n(E) = (\text{number vertices degree } k)k \text{ and,}$$

$$n(S) = \sum_i (\text{number vertices degree } i)i.$$

Since  $\rho_k(t)$  is the probability that a degree  $k$  vertex is infected,  $n(E)\rho_k(t)$  is the number of half edges leading to an infected degree  $k$  node. The probability that a half edge leads to an infected degree  $k$  node is then

$$P(E)\rho_k(t) = \frac{n(E)\rho_k(t)}{n(S)}.$$

Dividing numerator and denominator by  $N$  we get

$$P(E)\rho_k(t) = \frac{kP(k)\rho_k(t)}{\sum_i iP(i)}.$$

Summing over all the degrees in  $\mathcal{G}$  we find that the probability that a half edge leads to an infected node at time  $t$  is

$$\sum_k \frac{kP(k)\rho_k(t)}{\langle k \rangle}.$$

The probability that an edge leads to an infected node is the same as the probability a half edge leads to an infected node proving the claim.  $\square$

**Lemma B.2.** *The system (16) has a unique solution, which is given by*

$$\rho(t) = \begin{cases} \frac{\rho^0 \rho^1}{\rho^0 + (\rho^1 - \rho^0) e^{-(k\lambda-1)t}} & \text{if } k\lambda \neq 1 \\ \frac{\rho^0}{t\rho^0 + 1} & \text{if } k\lambda = 1, \end{cases}$$

where  $\rho^1 = 1 - \frac{1}{k\lambda}$ .

PROOF. When  $k\lambda = 1$ , the differential equation is  $\frac{d\rho}{dt} = -\rho(t)^2$ , which can be put in the form

$$-\frac{d\rho}{\rho(t)^2} = dt.$$

Integrating both sides we get  $\rho(t)^{-1} = t + \text{constant}$ . Solving for  $\rho(t)$  and the constant one easily verifies the result. Assume now that  $k\lambda \neq 1$ . Rewriting the differential equation as

$$\frac{d\rho}{-\rho(t) + k\lambda(1 - \rho(t))\rho(t)} = dt$$

and using partial fractions, we see that

$$\left[ \frac{1}{\rho(t)} + \frac{k\lambda}{k\lambda - 1 - k\lambda\rho(t)} \right] d\rho = (k\lambda - 1) dt.$$

Integrating both sides we get

$$\ln(\rho(t)) - \ln(k\lambda - 1 - k\lambda\rho(t)) = (k\lambda - 1)t + \text{constant}.$$

Exponentiating both sides we obtain, for some constant  $C$ ,

$$\frac{\rho(t)}{k\lambda - 1 - k\lambda\rho(t)} = C e^{(k\lambda-1)t}.$$

Finally, solving for  $\rho(t)$ , then using the initial condition  $\rho(0) = \rho^0$  and following some algebra, we get our result.  $\square$

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